On the Finiteness of the $\mathcal{N} = 4$ SUSY Nonlinear Sigma Model in Three Dimensions

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Abstract

We construct the $\mathcal{N} = 4$ supersymmetric nonlinear sigma model in three dimensions which can be expanded in $1/N$. We evaluate the effective action at leading order in the $1/N$ expansion and show the finiteness of the model to this order.

1 Introduction

Supersymmetric field theories have softer ultraviolet behavior than non-supersymmetric theories. The structure of ultraviolet divergence cancellations in supersymmetric field theories including non-renormalizable theories is reviewed in Ref. [1]. In renormalizable theories, the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions and the $\mathcal{N} = 4$ supersymmetric nonlinear sigma model in two dimensions were found to be finite to all orders in perturbation theory. In non-renormalizable theories, the maximal super Yang-Mills theories in $d = 5$, 6, and 7 were found to be divergent at four, three, and two loops respectively.

Three-dimensional nonlinear sigma models are perturbatively non-renormalizable, but they are argued to be renormalizable in the $1/N$ expansion [2, 3, 4, 5]. The $O(N)$ and CP^{N-1} sigma models were studied to next-to-leading order in $1/N$ and their β-functions were determined to this order [5, 6]. Supersymmetry appears to play a role in controlling the β-function through UV divergences in this class of field theories. In the $\mathcal{N} = 1$ supersymmetric $O(N)$ nonlinear sigma model, the next-to-leading order term in the β-function (the part due to logarithmic divergences) turned out to be absent [7]. In the $\mathcal{N} = 2$ supersymmetric CP^{N-1} sigma model, the next-to-leading order term in the β-function (due to both logarithmic and power divergences) was found to vanish [8, 9]. These results are reminiscent of the UV properties of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions.

It is very interesting to study whether the three-dimensional nonlinear sigma model with higher (i.e., $\mathcal{N} = 4$) extended supersymmetry has better UV properties. In this view we study the UV properties of the $\mathcal{N} = 4$ supersymmetric nonlinear sigma model in three dimensions using the $1/N$ expansion. One model of this kind which comes to our mind is the supersymmetric HP^{N-1} model (an extension of $O(N)$ and CP^{N-1} models), but this model is known to be a consistent model only when it is coupled to supergravity [10]. In four dimensions we have the $\mathcal{N} = 2$ supersymmetric nonlinear sigma model employing the cotangent bundle of CP^{N-1} [11, 12]. This model can be used to construct the $\mathcal{N} = 4$ supersymmetric nonlinear sigma model in three dimensions which serves the purpose of our $1/N$ study. In this letter we evaluate the effective action at leading order in $1/N$ and find that the model is finite to this order.

2 The $\mathcal{N} = 4$ SUSY Nonlinear Sigma Model in Three Dimensions

We start by considering the $\mathcal{N} = 2$ supersymmetric nonlinear sigma model in four dimensions. This model has been constructed both in $\mathcal{N} = 1$ superfields [12] and in the component language [11]. We follow the work of Curtright and Freedman [11]. The model has $2N$ complex scalar fields ϕ_i^α and N Dirac fields ψ^α, the superpartnars of ϕ_i^α, where $i = 1, 2$ and $\alpha = 1, \ldots, N$. These fields are subject to the constraints

$$\bar{\phi}_i^\alpha (\sigma_I)_{ij} \phi_j^\alpha = b_I, \quad I = 1, 2, 3, \tag{2.1}$$

$$R(\phi_i^\alpha \psi^{C\alpha} - \epsilon_{ij} \bar{\phi}_j^\alpha \psi^\alpha) + L(\bar{\phi}_i^\alpha \psi^\alpha - \epsilon_{ij} \phi_j^\alpha \psi^{C\alpha}) = 0, \tag{2.2}$$

where σ_I are the Pauli matrices and b_I is a fixed constant vector. $R, L = \frac{1}{2}(1 \pm \Gamma_5)$ are the projection operators and $\epsilon_{ij} = \frac{1}{2}[(-)^j - (-)^i]$. In this letter we use Γ's for the Dirac matrices in four dimensions. The action is given by

$$S = \int d^4x [\overline{D_\mu \phi_i^\alpha} D^\mu \phi_i^\alpha + i \bar{\psi}^\alpha \Gamma_\mu D^\mu \psi^\alpha + \rho \bar{\psi}^\alpha \psi^\alpha + i\sigma \bar{\psi}^\alpha \Gamma_5 \psi^\alpha - (\rho^2 + \sigma^2) \bar{\phi}_i^\alpha \phi_i^\alpha], \tag{2.3}$$

where $D_\mu = \partial_\mu + iA_\mu$, and ρ, σ, and A_μ are real auxiliary fields. The target space of the model is known to be a Calabi manifold with hyper-Kähler metric on the cotangent bundle of CP^{N-1} [10, 13].

The action and the constraints are invariant under the pair of supersymmetry transformations

$$\delta\phi_i^\alpha = \bar{\epsilon}_i L \psi^\alpha - \epsilon_{ij} \bar{\epsilon}_j R \psi^\alpha, \tag{2.4}$$

$$\delta\psi^\alpha = (-i\Gamma_\mu D^\mu + \rho - i\Gamma_5\sigma)(R\phi_i^\alpha \epsilon_i - L\epsilon_{ij}\phi_i^\alpha \epsilon_j), \tag{2.5}$$

provided we use the on-shell values for the auxiliary fields ρ, σ and A_μ. Here ϵ_i are Majorana spinor parameters. The supersymmetry transformations of the auxiliary fields need not be specified in the present on-shell formulation.

We construct the $\mathcal{N} = 4$ supersymmetric nonlinear sigma model in three dimensions by dimensional reduction of the four-dimensional $\mathcal{N} = 2$ model considered above. We express the four-dimensional Dirac matrices as

$$\Gamma_\mu = \gamma_\mu \otimes \sigma_2, \quad \mu = 0, 1, 2, \quad \Gamma_3 = i \otimes \sigma_3, \quad \Gamma_5 = -1 \otimes \sigma_1, \tag{2.6}$$

where γ_μ are the Dirac matrices in three dimensions and they are given by $\gamma_0 = \sigma_2$, $\gamma_1 = i\sigma_3$ and $\gamma_2 = i\sigma_1$. We write the four-component Dirac spinors in four dimensions as

$$\psi^\alpha = \begin{pmatrix} \psi_1^\alpha \\ i\psi_2^\alpha \end{pmatrix}, \qquad \psi^{C\alpha} = \begin{pmatrix} \psi_1^{*\alpha} \\ i\psi_2^{*\alpha} \end{pmatrix}, \tag{2.7}$$

where ψ_1^α and ψ_2^α are two-component complex spinors in three dimensions. We substitute (2.6) and (2.7) into (2.2) and (2.3) and pick up only the zero mode (with respect to the third space coodinate) parts of the fields. We obtain the three-dimensional action

$$
\begin{aligned}
S &= \int d^3x [\overline{D_\mu \phi_i^\alpha} D^\mu \phi_i^\alpha + i\bar\psi_i^\alpha \gamma_\mu D^\mu \psi_i^\alpha + \tau \bar\psi_1^\alpha \psi_2^\alpha + \bar\tau \bar\psi_2^\alpha \psi_1^\alpha - \sigma(\bar\psi_1^\alpha \psi_1^\alpha - \bar\psi_2^\alpha \psi_2^\alpha) \\
&\quad - (\bar\tau\tau + \sigma^2)\bar\phi_i^\alpha \phi_i^\alpha],
\end{aligned}
\tag{2.8}
$$

where $\tau = \rho - iA_3$ is a complex scalar field. As for the constraints, the direction of b_I is immaterial, and we choose the bosonic constraint (2.1) to be $b_I = (0,0,N/g)$, where g is the coupling constant of the model. The costraints now read

$$\bar\phi_1^\alpha \phi_1^\alpha - \bar\phi_2^\alpha \phi_2^\alpha = N/g, \quad \bar\phi_1^\alpha \phi_2^\alpha = 0, \tag{2.9}$$

$$\bar\phi_1^\alpha \psi_1^\alpha - i\phi_2^\alpha \psi_2^{*\alpha} = 0, \quad \bar\phi_1^\alpha \psi_2^\alpha + i\phi_2^\alpha \psi_1^{*\alpha} = 0. \tag{2.10}$$

The supersymmetry transformations in three dimensions can be obaind by dimensional reduction of the supersymmetry transformations (2.4) and (2.5). To this end we decompose the two four-component Majorana spinor parameters ϵ_i into four two-component Majorana spinor parameters ϵ_i^1 and ϵ_i^2 in the same way as (2.7). The three-dimensional supersymmetry transformations are then given by

$$\delta\phi_i^\alpha = -\frac{1}{2}i[\bar\zeta_i^*(\psi_1^\alpha + i\psi_2^\alpha) + \epsilon_{ij}\bar\zeta_j(\psi_1^\alpha - i\psi_2^\alpha)], \tag{2.11}$$

$$\delta\psi_1^\alpha = \frac{1}{2}[(\gamma_\mu D^\mu - i\sigma + \tau)\phi_i^\alpha \zeta_i^* + (\gamma_\mu D^\mu - i\sigma - \tau)\epsilon_{ij}\phi_i^\alpha \zeta_j], \tag{2.12}$$

$$\delta\psi_2^\alpha = -\frac{1}{2}i[(\gamma_\mu D^\mu + i\sigma - \bar\tau)\phi_i^\alpha \zeta_i^* - (\gamma_\mu D^\mu + i\sigma + \bar\tau)\epsilon_{ij}\phi_i^\alpha \zeta_j], \tag{2.13}$$

where $\zeta_i = \epsilon_i^1 + i\epsilon_i^2$ are complex spinor parameters. The supersymmetry transformations of the auxiliary fields τ, σ and A_μ need not be specified for the same reason as given previously.

We have derived the three-dimensional $\mathcal{N} = 4$ supersymmetric nonlinear sigma model (2.8) with (2.9) and (2.10). Introducing the Lagrange multiplier fields α, β, c and e, the

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Euclidean action is written as

$$
\begin{aligned}
S &= \int d^3x [\overline{D_\mu \phi_i^\alpha} D_\mu \phi_i^\alpha + i\bar\psi_i^\alpha \gamma_\mu D_\mu \psi_i^\alpha - \tau\bar\psi_1^\alpha \psi_2^\alpha - \bar\tau\bar\psi_2^\alpha \psi_1^\alpha + \sigma(\bar\psi_1^\alpha \psi_1^\alpha - \bar\psi_2^\alpha \psi_2^\alpha) \\
&\quad + (\bar\tau\tau + \sigma^2)\bar\phi_i^\alpha \phi_i^\alpha - \alpha(\bar\phi_1^\alpha \phi_1^\alpha - \bar\phi_2^\alpha \phi_2^\alpha - N/g) - \beta\bar\phi_1^\alpha \phi_2^\alpha - \bar\beta\bar\phi_2^\alpha \phi_1^\alpha + \bar\phi_1^\alpha \bar c \psi_1^\alpha + \phi_1^\alpha \bar\psi_1^\alpha c \\
&\quad + i\bar\phi_2^\alpha \bar c^* \psi_2^\alpha - i\phi_2^\alpha \bar\psi_2^\alpha c^* + \bar\phi_1^\alpha \bar e \psi_2^\alpha + \phi_1^\alpha \bar\psi_2^\alpha e - i\bar\phi_2^\alpha \bar e^* \psi_1^\alpha + i\phi_2^\alpha \bar\psi_1^\alpha e^*].
\end{aligned}
\tag{2.14}
$$

The model contains seven kinds of auxiliary fields: a $U(1)$ vector A_μ, two complex scalars τ, β, two real scalars σ, α, and two complex spinors c, e.

3 The Leading Order

Integrating over the fields ϕ_i^α, $\bar\phi_i^\alpha$, ψ_i^α and $\bar\psi_i^\alpha$, we obtain from (2.14) the effective action

$$
S_{eff} = N\mathrm{Tr}\ln\Delta_{B1} + N\mathrm{Tr}\ln\Delta_{B2} - N\mathrm{Tr}\ln\Delta_{F1} - N\mathrm{Tr}\ln\Delta_{F2} + \frac{N}{g}\int d^3x\, \alpha + \ldots, \tag{3.1}
$$

where

$$
\begin{aligned}
\Delta_{F1} &= i\gamma_\mu D_\mu + \sigma, \tag{3.2}\\
\Delta_{F2} &= i\gamma_\mu D_\mu - \sigma - \bar\tau\Delta_{F1}^{-1}\tau, \tag{3.3}\\
\Delta_{B1} &= -D_\mu D_\mu + \bar\tau\tau + \sigma^2 - \alpha - \bar c\Delta_{F1}^{-1}c - (\bar e + \bar c\Delta_{F1}^{-1}\tau)\Delta_{F2}^{-1}(e + \bar\tau\Delta_{F1}^{-1}c) \tag{3.4}\\
\Delta_{B2} &= -D_\mu D_\mu + \bar\tau\tau + \sigma^2 + \alpha - \bar e^*\Delta_{F1}^{-1}e^* - (\bar c^* - \bar e^*\Delta_{F1}^{-1}\tau)\Delta_{F2}^{-1}(c^* - \bar\tau\Delta_{F1}^{-1}e^*)\\
&\quad - [\bar\beta - i\bar e^*\Delta_{F1}^{-1}c + i(\bar c^* - \bar e^*\Delta_{F1}^{-1}\tau)\Delta_{F2}^{-1}(e + \bar\tau\Delta_{F1}^{-1}c)]\Delta_{B1}^{-1}[\beta + i\bar c\Delta_{F1}^{-1}e^*\\
&\quad - i(\bar e + \bar c\Delta_{F1}^{-1}\tau)\Delta_{F2}^{-1}(c^* - \bar\tau\Delta_{F1}^{-1}e^*)]. \tag{3.5}
\end{aligned}
$$

Setting all fields to constants, we obtain the effective potential

$$
\begin{aligned}
\frac{V}{N} &= \int \frac{d^3k}{(2\pi)^3}\Big[\ln(k^2 + \langle\bar\tau\rangle\langle\tau\rangle + \langle\sigma\rangle^2 - \langle\alpha\rangle)\\
&\quad + \ln\Big(k^2 + \langle\bar\tau\rangle\langle\tau\rangle + \langle\sigma\rangle^2 + \langle\alpha\rangle + \frac{\langle\bar\beta\rangle\langle\beta\rangle}{k^2 + \langle\bar\tau\rangle\langle\tau\rangle + \langle\sigma\rangle^2 - \langle\alpha\rangle}\Big)\\
&\quad - \mathrm{tr}\ln(-\slashed{k} + \langle\sigma\rangle) - \mathrm{tr}\ln\Big(-\slashed{k} - \langle\sigma\rangle - \frac{\langle\bar\tau\rangle\langle\tau\rangle}{-\slashed{k} + \langle\sigma\rangle}\Big)\Big]\\
&\quad + (\langle\bar\tau\rangle\langle\tau\rangle + \langle\sigma\rangle^2)(\bar v_1 v_1 + \bar v_2 v_2) - \langle\alpha\rangle(\bar v_1 v_1 - \bar v_2 v_2 - 1/g)\\
&\quad - \langle\beta\rangle\bar v_1 v_2 - \langle\bar\beta\rangle\bar v_2 v_1\\
&= \int \frac{d^3k}{(2\pi)^3}[\ln((k^2 + \langle\bar\tau\rangle\langle\tau\rangle + \langle\sigma\rangle^2)^2 - \langle\alpha\rangle^2 + \langle\bar\beta\rangle\langle\beta\rangle) - \ln(k^2 + \langle\bar\tau\rangle\langle\tau\rangle + \langle\sigma\rangle^2)^2]\\
&\quad + (\langle\bar\tau\rangle\langle\tau\rangle + \langle\sigma\rangle^2)(\bar v_1 v_1 + \bar v_2 v_2) - \langle\alpha\rangle(\bar v_1 v_1 - \bar v_2 v_2 - 1/g)\\
&\quad - \langle\beta\rangle\bar v_1 v_2 - \langle\bar\beta\rangle\bar v_2 v_1, \tag{3.6}
\end{aligned}
$$

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where $v_i = \langle\phi_i^N\rangle/\sqrt{N}$. The vacuum expectation values of fields which are not in (3.6) have been set to zero.

The vacuum of the model is fixed by the saddle point conditions

$$\frac{1}{N}\frac{\partial V}{\partial\langle\tau\rangle} = \langle\bar{\tau}\rangle(\bar{v}_1 v_1 + \bar{v}_2 v_2) = 0, \tag{3.7}$$

$$\frac{1}{N}\frac{\partial V}{\partial\langle\sigma\rangle} = 2\langle\sigma\rangle(\bar{v}_1 v_1 + \bar{v}_2 v_2) = 0, \tag{3.8}$$

$$\frac{1}{N}\frac{\partial V}{\partial\langle\alpha\rangle} = -2\langle\alpha\rangle\int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2 + \langle\bar{\tau}\rangle\langle\tau\rangle + \langle\sigma\rangle^2)^2} - \left(\bar{v}_1 v_1 - \bar{v}_2 v_2 - \frac{1}{g}\right) = 0, \tag{3.9}$$

$$\frac{1}{N}\frac{\partial V}{\partial\langle\beta\rangle} = \langle\bar{\beta}\rangle\int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2 + \langle\bar{\tau}\rangle\langle\tau\rangle + \langle\sigma\rangle^2)^2} - \bar{v}_1 v_2 = 0, \tag{3.10}$$

$$\frac{1}{N}\frac{\partial V}{\partial v_1} = (\langle\bar{\tau}\rangle\langle\tau\rangle + \langle\sigma\rangle^2 - \langle\alpha\rangle)\bar{v}_1 = 0, \tag{3.11}$$

$$\frac{1}{N}\frac{\partial V}{\partial v_2} = (\langle\bar{\tau}\rangle\langle\tau\rangle + \langle\sigma\rangle^2 + \langle\alpha\rangle)\bar{v}_2 = 0. \tag{3.12}$$

We look for the supersymmetric vacuum and have set $-\langle\alpha\rangle^2 + \langle\bar{\beta}\rangle\langle\beta\rangle = 0$. The solution of (3.7) through (3.12) is

$$\bar{v}_1 v_1 = 1/g, \quad v_2 = 0, \quad \langle\tau\rangle = \langle\sigma\rangle = \langle\alpha\rangle = \langle\beta\rangle = 0. \tag{3.13}$$

The saddle point conditions (3.9) and (3.10) contain the infrared divergences when $\langle\tau\rangle = \langle\sigma\rangle = 0$. In deriving the solution (3.13) we have introduced the infrared cutoff. Since the ϕ_1^α fields have a vacuum expectation value, $SU(N)$ symmetry of the action is broken. The model has only the broken phase. Performing the shift

$$\phi_1^N \rightarrow \phi_1^N + \sqrt{N}v_1 \tag{3.14}$$

in (2.14) and calculating the effective action, we obain

$$\begin{aligned}
S_{eff} = {} & N\text{Tr}\ln\Delta_{B1} + N\text{Tr}\ln\Delta_{B2} - N\text{Tr}\ln\Delta_{F1} - N\text{Tr}\ln\Delta_{F2} \\
& + N\bar{v}_1 v_1\int d^3x(A_\mu A_\mu + A_\mu\partial_\mu\Delta_{B1}^{-1}\partial_\nu A_\nu + \bar{\tau}\tau + \sigma^2 - \alpha\Delta_{B1}^{-1}\alpha - \beta\Delta_{B2}^{-1}\bar{\beta} \\
& - \bar{c}\Delta_{F1}^{-1}c - \bar{e}\Delta_{F2}^{-1}e) + \frac{N}{g}\int d^3x\ \alpha + \dots. \tag{3.15}
\end{aligned}$$

The two-point functions of the auxiliary fields are obtained by evaluating functional derivatives of the effective action (3.15). All of these functions are finite because of cancellations between boson and fermion loops. This is in accord with the renormalizabiliaty

of the model. The effective propagators of the auxiliary fields are given by

$$D_{\mu\nu}^A(p) = \frac{1}{N}\frac{4}{\sqrt{p^2 + 8\bar{v}_1 v_1}}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right),$$

$$D^\tau(p) = \frac{1}{N}\frac{8}{\sqrt{p^2 + 8\bar{v}_1 v_1}}, \quad D^\sigma(p) = \frac{1}{N}\frac{4}{\sqrt{p^2 + 8\bar{v}_1 v_1}},$$

$$D^\alpha(p) = -\frac{1}{N}\frac{4p^2}{\sqrt{p^2 + 8\bar{v}_1 v_1}}, \quad D^\beta(p) = -\frac{1}{N}\frac{8p^2}{\sqrt{p^2 + 8\bar{v}_1 v_1}},$$

$$D^c(p) = \frac{1}{N}\frac{8\not{p}}{\sqrt{p^2 + 8\bar{v}_1 v_1}}, \quad D^e(p) = \frac{1}{N}\frac{8\not{p}}{\sqrt{p^2 + 8\bar{v}_1 v_1}}. \tag{3.16}$$

We note that the mixing terms between A_μ and α, σ vanish in the effective action (3.15) as those in the supersymmetric CP^{N-1} sigma model in three dimensions [9]. In the present model the fields ψ_i^α remain massless, so the term which involves $\epsilon_{\mu\nu\rho}$ is not induced in the two-point function of A_μ. Such a term is induced in the symmetric phase of the supersymmetric CP^{N-1} sigma model in three dimensions where the fermion fields become massive [9].

4   The β-Function and Finiteness

The saddle point conditions have turned out to be free from UV divergences. The coupling constant is not renormalized at leading order, so the β-function of the model is given by

$$\beta(\tilde{g}) = \tilde{g}, \tag{4.1}$$

where $\tilde{g}$ is the dimensionless coupling constant. The β-function has no fixed point and there is no phase transition. This β-function differs from that of the $O(N)$, CP^{N-1} sigma models and their supersymmetric versions. In these models only the coupling constant is renormalized at leading order, and the β-function of these models is given by $\beta(\tilde{g}) = \tilde{g}(1 - \tilde{g}/\tilde{g}_c)$, where $\tilde{g}_c$ is the critical point.

We have confirmed that the superficially divergent two- and three-point functions of the auxiliary fields are all finite at leading order. Therefore, the model is finite to leading order in $1/N$. This result is same as that of the two-dimensional model, where the supergraph techniques are used [12].

It is an important question whether the finiteness of the model persists to higher orders in $1/N$. This question can be invesigated to next-to-leading order in the same manner as the supersymmetric CP^{N-1} sigma model [9].

Acknowledgements

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